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Emergence of complexity in controlling simple regular networks

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Abstract – Quantifying the capacity of a given node or a bunch of nodes in maintaining a system's controllability is a crucial problem in complex networks and control theory. We give a systematic analysis of the ability of a single node or a pairs of nodes to control an undirected unweighted chain and ring. By combining algebraic theory and graph spectrum analysis, we derive analytic expressions for the control range of some given control inputs and find that complex phenomena emerge even from these simplest graph structures. Specifically, the control range is sensitive to the location of driver nodes and shows complex periodic behaviors. Our findings have implications for evaluating the control range and practically controlling complex networks.

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Introduction. – Controlling collective dynamics is the ultimate goal of investigating complex natural or technological systems, in which the interaction patterns among dynamical elements modeled as complex networks play crucial roles [1,2]. The past few years have witnessed the rapid development of control theory for complex networked systems [3], among which two representative theoretical frameworks are the structural control theory (SCT) [2] and the exact control theory (ECT) [4]. A great deal of efforts have been motivated by the development of the two control frameworks [5–17]. The central issue in network control is identifying a minimum set of driver nodes, by directly controlling which the network can be fully controlled to reach any target state under certain constraints. Practically, however, some driver node discerned by the control theories may be not externally accessible, rendering full control of a network from a minimum set of driver nodes impossible. On the other hand, in some real situations, it is only necessary to control a fraction of nodes rather than all nodes. Thus, an alternative question is how to determine which nodes are indirectly controllable by directly controlling a given set of externally accessible nodes. Despite some pioneer efforts dedicated to addressing this question [18,19], we still lack a general and theoretical approach available for any types of networks. Even for a very simple regular network, a full understanding of the control range in terms of directly controlling a small fraction of nodes is yet lacking.

In this letter, we explore the control range of two simple regular graphs —chain and ring— for a given set of external inputs. We offer theoretical predictions for the control range, *i.e.*, the number of controllable nodes for a given set of inputs at any locations. Among the nodes two types of interactions are considered which are characterized by the Laplacian matrix and the adjacency matrix, respectively. We find that complex phenomena emerge in these simple graphs. In particular, the control range is very sensitive to the location of imposed input signals and several periodic behaviors are observed. The analytical prediction based on eigenvector decomposition indicates that the control range is closely related to integer factoring and prime numbers, which accounts for the complex phenomena. Numerical results show an exact agreement with the analytical results for both regular graphs. Our findings gain insight into the complexity in controlling complex networks and are valuable for developing a general theoretical framework for evaluating control range and practically controlling complex networks.

Model and method. – Consider a controlled network that has N nodes, as described by the following linear ordinary differential equations [20]:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \tag{1}$$

where the vector $\mathbf{x} = (x_1, \ldots, x_N)^{\mathrm{T}}$ represents the state of each node at time t, $A = (a_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ denotes

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the coupling matrix of the system, B is the $N \times m$ control matrix and $\mathbf{u}(t)$ is the input signal.

According to Kalman's rank condition [21,22], system (1) is controllable if and only if the $N \times Nm$ controllability matrix $C = [B, AB, A^2B, \ldots, A^{N-1}B]$ has full rank. The control range R, defined as the rank of the controllability matrix C ($R \equiv \mathrm{rank}(C)$), specifies the dimension of the controllable subspace of the system [21,22]. In other words, control range R quantifies the number of nodes that the control matrix B can effectively control.

In this letter, our goal is to theoretically and numerically explore the control range R of simple regular graphs for some certain given inputs. To achieve this goal, we employ the Popov-Belevitch-Hautus (PBH) [23,24] test theory as an alternative to Kalman's condition to offer theoretical results for the control range of regular graph and reveal the emerging complexity. In particular, the PBH test specifies uncontrollable subspace in terms of eigenvalues and eigenvectors, which allows us to derive the dimension of controllable subspace analytically.

Specifically, according to PBH test theory, if the system (1) is not fully controllable, there exists an uncontrollable state vector α that satisfies the following condition:

$$\begin{aligned} (A^{\mathrm{T}} - \lambda I)\alpha &= 0, \\ B^{\mathrm{T}}\alpha &= 0, \end{aligned} \tag{2}$$

which means $\alpha \in V_\lambda \cap \mathrm{N}(B^{\mathrm{T}})$, where V_λ is the subspace of eigenvalue λ and $\mathrm{N}(B^{\mathrm{T}})$ is the null space of B^{T}. For a symmetric matrix A, any two eigenvectors of distinct eigenvalues are orthogonal [25,26].

It can be proven that the uncontrollable subspace is $V_\lambda \cap \mathrm{N}(B^{\mathrm{T}})$, which is the orthogonal complement to the controllable subspace [27,28]. Thus, we have

$$R = N - \dim(V_\lambda \cap \mathrm{N}(B^{\mathrm{T}})), \tag{3}$$

where dim represents the space dimension, which means the control range measurement R is the number of eigenvectors of A that violate eq. (2).

We use eq. (3) to explore the control range R of two simple regular networks —chain and ring— for some representative external inputs. It is noteworthy that although the controllability of the simple regular graphs has been intensively studied previously [4,27,29,30], the control range is still intact prior to our current work.

According to the ref. [4], the minimum number of driver nodes for fully controlling a chain graph and a ring graph are one and two, respectively. Thus, we consider a single input case for the chains and double input case for the rings and focus on how the locations of the imposed input signals affect the control range R. Below, we derive analytical results of R for chains and rings with different sizes and different locations of external inputs based on eq. (3).

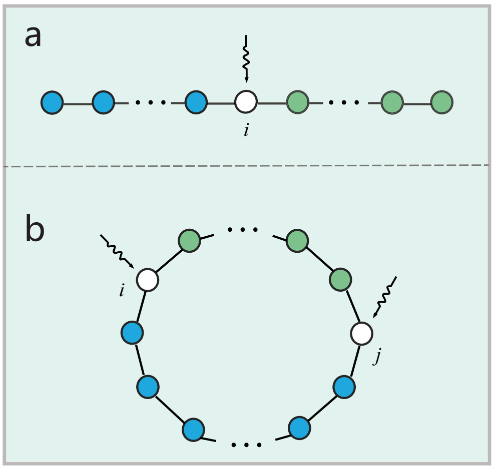

Fig. 1: (Color online) Illustration of network models and construction in this letter. (a) Chain graph with an external control input imposed on node i, represented as a wavy arrow. (b) Ring graph with node i and node j are both controlled. When $i = j$, the control inputs degenerate to one controller.

Chain. – The Laplacian matrix of a chain with N nodes has the following structure:

$$A_L = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \tag{4}$$

and all its eigenvalues are given as $\lambda_k = 2 - 2\cos[\pi(k-1)/N]$ for $k = 1, \ldots, N$ [31–33].

As shown in fig. 1(a), with a single control signal applied at node i, two subchains with size $i-1$ and $N-i$ are isolated from the chain, which is marked by blue and green, respectively. When we only directly control node i, the control matrix B reduces to a column vector b_i with a single non-zero entry. We can prove that the control range R can be calculated via

$$R_i = N - \{\mathrm{GCD}(2i-1, 2[N-i]+1) - 1\}/2, \tag{5}$$

where $\mathrm{GCD}(m,n)$ is the greatest common divisor of the two positive integers (m and n) (see appendix for the proof).

The numerical results of R_i for a chain of 30 nodes with different input locations are shown in fig. 2(a). Complex behavior emerges, as suggested by eq. (5). R is sensitive to the change of input location. A counterintuitive result is that the smallest value of R is not reached at the middle of the chain, *i.e.*, $i = 15$ or $i = 16$. In contrast, the smallest control range arises at $i = 8$ and $i = 23$, about a quarter and 3 quarters of the chain. There exist different levels of local minima as well, arising at specific locations. Although there is no intuitive explanation for the emerging phenomena from simple regular graph, the

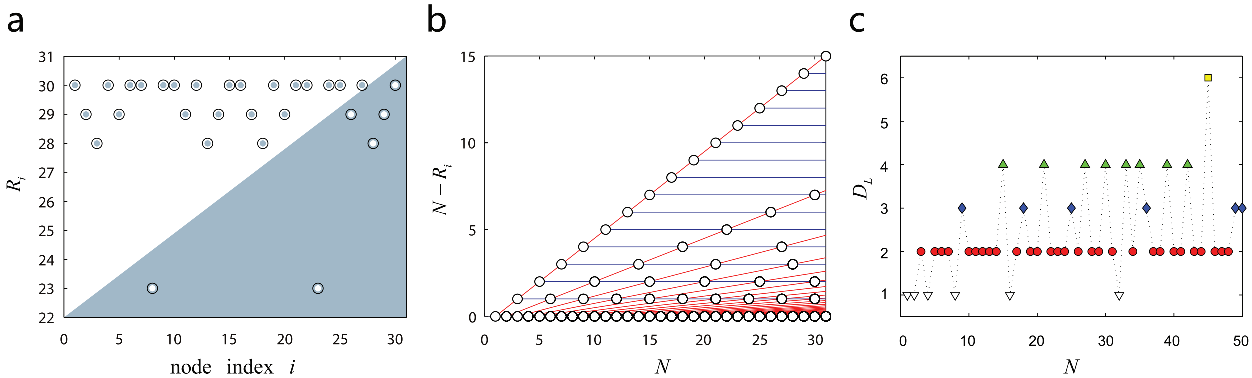

Fig. 2: (Color online) (a) Values of R_i for controlling each node in the Laplacian matrix of a chain with 30 nodes, the black circles are obtained from eq. (5) and the solid circles are obtained directly by calculating the rank (C). (b) All possible values of $N-R_i$ for the Laplacian matrix *vs.* the chain length N. For a fixed value of N, there are a finite number of R_i values. (c) D_L, the number of distinct values of control range R_i *vs.* N. Each distinct value of D_L is marked with a different symbol and color.

numerical findings are in exact agreement with analytical predictions. Figure 2(b) shows, two clusters of periodic behavior of $N-R_i$ present as N is increased. The periodic phenomena can be verified in terms of eq. (5).

Furthermore, we explore the number of possible values of control range of different system sizes. Specifically, let D_L denote the number of distinct values of control range. According to the eq. (5), we note that $2i-1, 2(N-i)+1$ are odd and $2i-1+2(N-i)+1=2N$, hence the D_L is determined by the odd integer factors of $2N$. As the symmetry of the chain, we only consider N for simplification. Let us denote p as the odd integer factors of N and let l_p represents the number of the odd integer factors. Thus, we can calculate N_L through the number of all the possible integer solutions satisfying the following equation:

$$N = l_p \cdot p \ (1 \leq p \leq N). \tag{6}$$

If N is the product of even numbers, there is only one integer solution of eq. (6): $l_p = N$ and $p=1$, leading to $D_L = 1$ (the hollow triangles in fig. 2(c)). When N is a prime number greater than 2 [34], $D_L = 2$, because the integer solutions are $(l_p, p) = (1, N)$ and $(l_p, p) = (N, 1)$, as part of the red circles in fig. 2(c). ∎

Next, we consider the control range of a chain characterized by an adjacency matrix. For a chain graph with N nodes, the adjacency matrix is

$$A_G = P_N = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}, \tag{7}$$

and its eigenvalues are $\lambda_k = 2\cos[\pi k/(N+1)]$ for $k = 1,\ldots,N$, and the algebraic multiplicity of each distinct eigenvalue [32,33] is 1. We can prove that by imposing a single input signal at location i,

$$R_i = (N+1) - \mathrm{GCD}(i, N+1-i). \tag{8}$$

The control range R_i of a chain with size 47 is shown in fig. 3(a). Figure 3(b) shows, the distribution of the control range values *vs.* the network size N.

Let D_A denote the number of distinct R_i in the case of adjacency matrices. Since $N+1 = i+(N-i+1)$, according to the eq. (10), D_A is determined by the integer factors of $N+1$. Let us denote $i = a*p$ and $N-i+1 = b*p$, where $f_p = a+b$ and p is $\mathrm{GCD}(i, N-i+1)$ ($p \leq \lfloor N/2 \rfloor + 1$). Then, D_A is the total number of possible integer solutions of the following equation:

$$N+1 = f_p \cdot p \ \left(1 \leq p \leq \left\lfloor \frac{N}{2} \right\rfloor + 1\right). \tag{9}$$

Thus, the solution of D_A is related with prime numbers [34]. Specifically, if $N+1$ is a prime number, $D_A = 1$, because the only one integer solution of eq. (9) is $f_p = N+1$ and $p=1$, as shown by the hollow triangles in fig. 3(c). When $N+1$ is the square of a prime number, there are two integer solutions: $(f_p, p) = (N+1, 1)$ and $(f_p, p) = (\sqrt{N+1}, \sqrt{N+1})$, accounting for $D_A = 2$ (the red circles in fig. 3(c)). ∎

Ring. – The Laplacian matrix of a ring graph with N nodes is

$$A_L = \begin{bmatrix} 2 & -1 & 0 & \ldots & -1 \\ -1 & 2 & -1 & \ldots & 0 \\ 0 & -1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & \ldots & 2 \end{bmatrix}. \tag{10}$$

Its eigenvalues are $\lambda_k = 2 - 2\cos(2\pi(k-1)/N)$ for $k = 1,\ldots,N$, showing that $\lambda_k = \lambda_{N-k+2}$ and most of the Laplacian eigenvalues have double multiplicity [32,33,35].

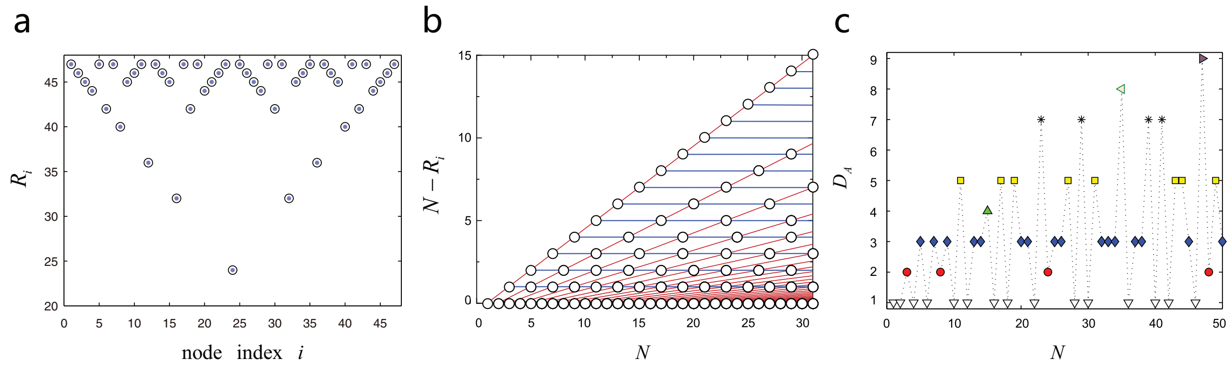

Fig. 3: (Color online) (a) Nodal control range R_i for the adjacency matrix of a chain with 47 nodes, where the black circles are obtained from eq. (8) and the solid circles are obtained by calculating rank (C). (b) All Possible values of $N - R_i$ for the adjacency matrix of chain of different length N. (c) D_A for the adjacency matrix of chain *vs.* system size N. Each distinct value of D_A is marked with a different symbol and color.

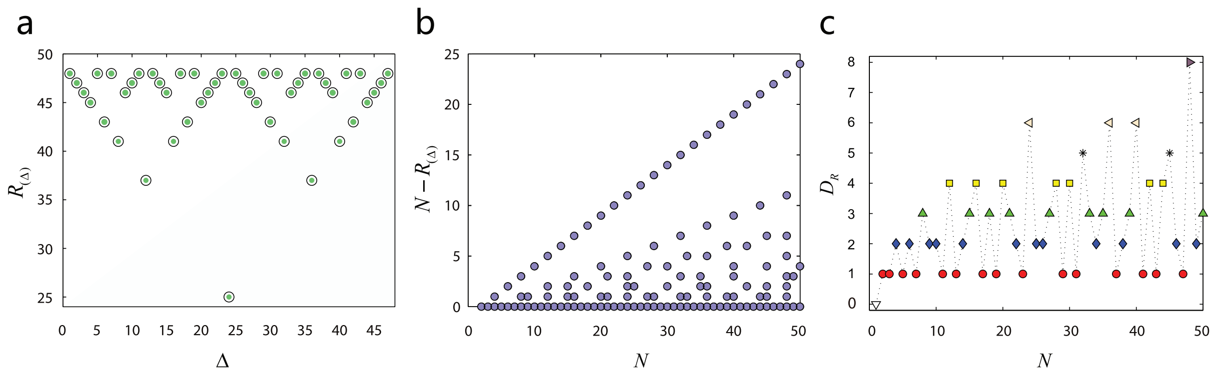

Fig. 4: (Color online) (a) $R_{(\Delta)}$ for the Laplacian matrix of the ring *vs.* Δ for a ring graph with size 48, where the black circles are obtained from eq. (12) and the solid circles are obtained from calculation of rank (C). (b) Distribution of the values of $N - R_{(\Delta)}$ for the Laplacian matrix of the ring *vs.* the system size N. (c) D_R, the number of distinct values of control range $R_{(\Delta)}$ for the ring *vs.* N. Each distinct value of D_R is marked with a different symbol and color.

As shown in the fig. 1(b), when we control both node i and node j ($i \leq j$, $\Delta \equiv j-i$), the control matrix B reduces to $B_{(i,j)} = [b_i, b_j]$, and the ring splits into two subchains with size $\Delta - 1$ (nodes in blue) and $N - \Delta - 1$ (nodes in green), respectively. Because of the rotational symmetry of ring, the control range $R_{(i,j)}$ is simply determined by the value of $\Delta = j - i$.

First, we prove that, for the control matrix $B_{(i,i)} = B_i = b_i$, we have

$$R_{(i,i)} = R_i = R_{(\Delta=0)} = \lfloor N/2 \rfloor + 1, \text{ for } i = 1, \ldots, N. \quad (11)$$

Next, we prove that, when controlling node i and node j ($i \neq j$), we have

$$R_{(i,j)} = R_{(\Delta)} = N - \sum_{k=1}^{\gcd(\Delta)-1} \eta_k, \quad (12)$$

where $\gcd(\Delta) = \mathrm{GCD}(\Delta, N - \Delta)$ and

$$\eta_k = \begin{cases} 1, & N \cdot k/\gcd(\Delta) \text{ is even}, \\ 0, & N \cdot k/\gcd(\Delta) \text{ is odd}. \end{cases} \quad (13)$$

The control range of different Δ, shown in fig. 4(a), exhibits complex phenomena. Figure 4(b) shows all the possible values of R_Δ for different system size. Let D_R denote the number of distinct $R_{(\Delta)}$ of the ring. The dependence of D_R on N is shown in fig. 4(c). ■

For the adjacency matrix A_G of ring graph ($A_G = 2I - A_L$), A_G has the same eigenvectors with A_L [32,33], thus the control range of controlling one node or two nodes are also given as eqs. (11) and (12), respectively.

Conclusions. – In sum, we study the control range of two simple regular graphs, chain and ring graphs, for a given set of external inputs, and offer rigorous theoretical predictions for the control range of both the Laplacian matrix and adjacency matrix. We find that the control range is quite related to the integer factoring and prime numbers, and shows special periodic behaviors. The presented results gain insight into the complexity in controlling complex networks and help us to develop a theorectical framework for evaluating control range.

Appendix. –

Proof of chain with Laplacian matrix. Because of the special structure of A_L, we firstly define a matrix $Q_\mu \in \mathbb{R}^{\mu\times\mu}$ as

$$Q_\mu = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \qquad \text{(A.1)}$$

whose μ distinct eigenvalues are $\lambda_k = 2 - 2\cos[(2k-1)\pi/(2\mu+1)]$ for $k = 1,\dots,\mu$ [32,33].

To satisfy eq. (2), the corresponding eigenvector $\mathbf{v}$ of A_L must be in the form $\mathbf{v} = [\mathbf{v}_1, 0, \mathbf{v}_2]^{\mathrm{T}}, \mathbf{v}_1 \in \mathbb{R}^{i-1}, \mathbf{v}_2 \in \mathbb{R}^{N-i}$. Component-wise eq. (2) is written as

$$\begin{aligned} Q_{i-1}\cdot \mathbf{v}_1 &= \lambda \mathbf{v}_1, \\ (\mathbf{v}_1)_{i-1} + (\mathbf{v}_2)_1 &= 0, \\ (\Pi Q_{N-i}\Pi)\cdot \mathbf{v}_2 &= \lambda \mathbf{v}_2, \end{aligned} \qquad \text{(A.2)}$$

where $(\mathbf{v}_1)_{i-1}$ is the $(i-1)$-th element of the vector $\mathbf{v}_1$ and $(\mathbf{v}_2)_1$ is the first element of the vector $\mathbf{v}_2$. $\Pi = \Pi^{\mathrm{T}} = \Pi^{-1}$ is the symmetric permutation matrix which reverses all the components of a vector. It is easily verified that (Q_{i-1}) and $(\Pi Q_{N-i}\Pi)$ have the same eigenvalue in the sense that Π is an orthogonal matrix [36,37].

According to the similar proof routines in ref. [30], to satisfy the first and third conditions in eq. (A.2), $\mathbf{v}_1$ and $\mathbf{v}_2$ should be the corresponding eigenvectors of λ for Q_{i-1} and Q_{N-i}, respectively. For the second condition, if Q_{i-1} and Q_{N-i} have at least one common eigenvalue, say λ_0, with corresponding eigenvectors $\mathbf{v}_{10}$ and $\mathbf{v}_{20}$, respectively. Then, we get $\mathbf{v} = [\mathbf{v}_{10}, 0, a\mathbf{v}_{20}]$ and $\lambda = \lambda_0$, which satisfy eq. (A.2), where a is a scaling factor given by $(\mathbf{v}_{10})_{i-1} + a(\mathbf{v}_{20})_1 = 0$.

We thus see that, if Q_{i-1} and Q_{N-i} have one common eigenvalue, there exists one corresponding eigenvector satisfying eq. (2). Note that the eigenvalues of Q_{i-1} and Q_{N-i} are given as $\lambda_m = 2 - 2\cos[\frac{(2m-1)\pi}{2i-1}]$ $(m = 1,\dots,i-1)$ and $\lambda_n = 2 - 2\cos[\frac{(2n-1)\pi}{2(N-i)+1}]$ $(n = 1,\dots,N-i)$, respectively. Thus the number of solutions for $\frac{(2m-1)}{2i-1} = \frac{(2n-1)}{2(N-i)+1}$ is given by $\{\mathrm{GCD}(2i-1, 2[N-i]+1) - 1\}/2$, correspondingly, there exist the same number of eigenvectors satisfying eq. (2). Thus, we obtain the eq. (5). ■

Proof of chain with adjacency matrix. When controlling node i only, the chain splits into two subchains with size $i-1$ and $N-i$, whose corresponding matrices are P_{i-1} and P_{N-i}, respectively. Following the similar proof of the Laplacian case, any eigenvector $\mathbf{v} = [\mathbf{v}_1, 0, \mathbf{v}_2] \in V_\lambda \cap N(B^{\mathrm{T}})$ associated with the eigenvalue λ satisfies $P_{i-1}\cdot \mathbf{v}_1 = \lambda\mathbf{v}_1$, $(\mathbf{v}_1)_{i-1} + (\mathbf{v}_2)_1 = 0$ and $P_{N-i}\cdot\mathbf{v}_2 = \lambda\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ with $(\mathbf{v}_1)_{i-1} + a(\mathbf{v}_2)_1 = 0$ is the corresponding eigenvector of P_{i-1} and P_{N-i}, respectively. Similarly, the number of the eigenvector $\mathbf{v}$ is one-to-one correspondence with the eigenvalue λ. Note that the eigenvalues of P_{i-1} and P_{N-i} are given by $\lambda_m = 2\cos(\pi m/i)$ (for $m = 1,\dots,i-1$) and $\lambda_n = 2\cos[\pi n/(N-i+1)]$ (for $n = 1,\dots,N-i$), respectively, hence the number of common eigenvalues is given by $\mathrm{GCD}(i, N-i+1) - 1$. Thus, eq. (8) is verified. ■

Proof: ring, Laplacian, single control. Without loss of generality, we impose a control signal at node 1. We first define a matrix M_μ as

$$M_\mu = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}. \qquad \text{(A.3)}$$

It can be easily verified that the eigenvalues of M_μ is

$$\lambda_k = 2 - 2\cos[\pi k/(\mu+1)] \text{ for } k = 1,\dots,\mu, \qquad \text{(A.4)}$$

and the corresponding eigenvector of λ_k is $(\mathbf{v}_k)_m = \sin[m\cdot k\cdot\pi/(\mu+1)]$ [30,32,33].

The corresponding eigenvector $\mathbf{v}$ of A_L that satisfies eq. (2) should be in the form $\mathbf{v} = [0, \mathbf{v}_k]$ and follows the constraints below:

$$\begin{aligned} (\mathbf{v}_k)_1 + (\mathbf{v}_k)_{N-1} &= 0, \\ (M_{N-1})\cdot \mathbf{v}_k &= \lambda_k \mathbf{v}_k, \end{aligned} \qquad \text{(A.5)}$$

where λ_k and $\mathbf{v}_k \in \mathbb{R}^{N-1}$ are the k-th eigenvalue and eigenvector of M_{N-1}, respectively. The first condition of eq. (A.5) shows that the first and last components of $\mathbf{v}_k$ have the same absolute value but opposite sign. As we know, $(\mathbf{v}_k)_1 = \sin[k\cdot\pi/N]$ and $(\mathbf{v}_k)_{N-1} = \sin[(N-1)\cdot k\cdot\pi/N] = \sin(k\pi - k\pi/N)$. According to the trigonometric formulas, we have: when k is odd,

$$(\mathbf{v}_k)_{N-1} = \sin[\pi - k\cdot\pi/N] = (\mathbf{v}_k)_1; \qquad \text{(A.6)}$$

when k is even,

$$(\mathbf{v}_k)_{N-1} = \sin[2\pi - k\cdot\pi/N] = -(\mathbf{v}_k)_1. \qquad \text{(A.7)}$$

Clearly, the necessary and sufficient condition for $(\mathbf{v}_k)_1 + (\mathbf{v}_k)_{N-1} = 0$ is k is even. Thus, there are $\lfloor (N-1)/2 \rfloor$ pairs of eigenvalue and eigenvector satisfying eq. (A.5). Therefore, we have $R_{(\Delta=0)} = N - \lfloor (N-1)/2\rfloor = \lfloor N/2 \rfloor + 1$. ■

Proof: ring, Laplacian, control two nodes. Without loss of generality, let $i = 1, j = \Delta + 1$. Then, any of the nonzero solution vectors $\mathbf{v} \in \mathbb{R}^N$ of A_L satisfying eq. (2) should be in the form $\mathbf{v} = [0, \mathbf{v}_{k_1}, 0, \mathbf{v}_{k_2}]$ and satisfying

$$\begin{aligned} &(\mathbf{v}_{k_1})_1 + (\mathbf{v}_{k_2})_{N-\Delta-1} = 0, \\ &M_{\Delta-1}\cdot \mathbf{v}_{k_1} = \lambda_{k_1}\mathbf{v}_{k_1}, \\ &(\mathbf{v}_{k_1})_{\Delta-1} + (\mathbf{v}_{k_2})_1 = 0, \\ &(M_{N-\Delta-1})\cdot \mathbf{v}_{k_2} = \lambda_{k_2}\mathbf{v}_{k_2}, \end{aligned} \qquad \text{(A.8)}$$

where $(\lambda_{k_1}, \mathbf{v}_{k_1})$ $(\mathbf{v}_{k_1} \in \mathbb{R}^{\Delta-1})$ and $(\lambda_{k_2}, \mathbf{v}_{k_2})$ $(\mathbf{v}_{k_2} \in \mathbb{R}^{N-\Delta-1})$ are the k_1-th and k_2-th eigenvalue and eigenvector pairs of $M_{\Delta-1}$ and $M_{N-\Delta-1}$, respectively. For the second and fourth conditions in eq. (A.8), there must exist common eigenvalues shared by $M_{\Delta-1}$ and $M_{N-\Delta-1}$. According to eq. (A.4), the number of common eigenvalues is $\mathrm{GCD}(\Delta, N-\Delta) - 1 = \gcd(\Delta) - 1$.

Because of the special relations between the first entry and last entry of $\mathbf{v}_{k_1}$ (and $\mathbf{v}_{k_2}$) shown in eqs. (A.6) and (A.7), to satisfy the first and third conditions of eq. (A.8), the entries of $\mathbf{v}_{k_1}$ and $\mathbf{v}_{k_2}$ should follow $(\mathbf{v}_{k_1})_1 = (\mathbf{v}_{k_1})_{\Delta-1}$ and $(\mathbf{v}_{k_2})_1 = (\mathbf{v}_{k_2})_{N-\Delta-1}$ or $(\mathbf{v}_{k_1})_1 = -(\mathbf{v}_{k_1})_{\Delta-1}$ and $(\mathbf{v}_{k_2})_1 = -(\mathbf{v}_{k_2})_{N-\Delta-1}$, which means k_1 and k_2 should both be even or odd, or equivalently, $(k_1 + k_2) = \frac{\Delta \cdot k}{\gcd(\Delta)} + \frac{(N-\Delta)\cdot k}{\gcd(\Delta)} = N \cdot k/\gcd(\Delta)$ is even, where k represents the k-th common eigenvalue. For the sake of clarity, we define η_k as eq. (13). Thus, combining all the conditions, we have eq. (12). ■

* * *

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